III. "On a Method of Destroying the Effects of slight Errors of Adjustment in Experiments of Changes of Refrangibility due to Relative Motions in the Line of Sight." By E. J. STONE, F.R.S., Director of the Radcliffe Observatory, Oxford. Received January 17, 1881.

Let arrangements be made for the reversion of the prisms without any disturbance of the other optical arrangements, including, of course, the position of the cylindrical lens, if one be used. Any slight errors of adjustment which prevent the light from the star and the comparison light from falling upon the train of prisms under the same optical circumstances, so far as mere direction is concerned, will have opposite effects in the reversed positions of the prisms; but the separation of the emergent lights due to relative motion will remain unchanged by the reversal of the positions of the prisms.

If, therefore, the apparent change of refrangibility due to relative motion remains unchanged by the reversion of the prisms, all doubts about the effects of errors of adjustment will be removed. But if the results in the reversed positions of the prisms sensibly differ, then the existing errors of adjustment must be removed, or their effects allowed for by taking a mean of the results in reversed positions, before any reliance can be fairly placed upon the determination of relative motions in the line of sight.

A reversible spectroscope was arranged by me, and made by Mr. Simms, some years ago, but I have never since had an equatoreal, with a good driving clock, under my control with which the experiment indicated could be properly tried.

With the direct prisms now in use, the required reversion can be easily arranged. I am not likely, for some time, to have the use of a good equatoreal, and I, therefore, publish the plan with the hope that some one more fortunately situated may give it a fair trial.

The experiment is a crucial one, and, in my opinion, should be tried.

IV. "On an Improved Bimodular Method of computing Natural and Tabular Logarithms and Anti-Logarithms to Twelve or Sixteen Places, with very brief Tables." By ALEXANDER J. ELLIS, B.A., F.R.S., F.S.A. Received January 17, 1881.

SECTION I.—NATURE OF THE BIMODULAR METHOD AND ITS IMPROVEMENT.

The *Bimodulus* is a constant, which is exactly double of the modulus of any system of logarithms. The *Bimodular Method* is derived from

the familiar proposition that, when the difference of two numbers is small, the difference of their logarithms is nearly equal to the bimodulus multiplied by the difference and divided by the sum of the numbers themselves. The improvement here for the first time effected, consists in prefixing a brief *preparation*, which makes the method universally applicable, and subjoining an easy *correction* depending on the transformation of a well-known series proceeding by the odd powers of the difference divided by the sum of two numbers, whereby the number of places obtained is greatly increased. This method is here applied for finding the natural and tabular logarithms of any number to twelve places of figures by means of a table of two pages for each kind of logarithm, and to sixteen places by help of a seven-place table of tabular or Briggs's logarithms. An extremely simple rule, which, so far as I know, was never before imagined, enables us to pass from the logarithm to the number, that is, to find anti-logarithms from the same tables. Although the method is applicable to any system of logarithms, and was actually first applied by me to the direct calculation of musical logarithms to the bases 2 (octave), $\sqrt[12]{2}$ (equal semitone), and $81 \div 80$ (comma), and appropriate tables have been constructed, I confine myself for brevity to natural and tabular logarithms. The tables are constructed from existing materials, but the method is capable of constructing them independently.

Section II.—Principles of the Bimodular Method and its Improvement.

Fundamental Relations.—Let n and d be any whole numbers of which d is the smaller, and let $p = d \div n$, a proper fraction. Let

$$\text{nat. log}\,(1+p) = y, \text{ and } \log\,(1+p) = \mathrm{M}y \quad . \quad . \quad . \quad (1)$$

where M is the modulus, and hence 2M the bimodulus to any unspecified system of logarithms marked by log. Let

$$\frac{d}{2n+d} = \frac{p}{2+p} = q,\ 2q = x,\ 2\mathrm{M}q = \mathrm{M}x = z \quad . \quad . \quad . \quad . \quad (2).$$

In future n and $n+d$ will often be called "the numbers," n "the tabular number," d "the difference," $2\mathrm{M}d$ "the dividend," $2n+d$ "the sum" or "divisor," and $2\mathrm{M}d \div (2n+d)$ "the quotient."

Now it is familiarly known that

$$y = p - \tfrac{1}{2}p^2 + \tfrac{1}{3}p^3 - \tfrac{1}{4}p^4 + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$= 2(q + \tfrac{1}{3}q^3 + \tfrac{1}{5}q^5 + \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Putting in (4) the values of q in terms of x and z from (2) we have

$$y = x + \tfrac{1}{12}x^3 + \tfrac{1}{80}x^5 + \tfrac{1}{448}x^7 + \ldots = x + c \quad . \quad . \quad . \quad . \quad (5),$$

$$My = z + \tfrac{1}{12} \cdot \frac{z^3}{M^2} + \tfrac{1}{80} \cdot \frac{z^5}{M^4} + \tfrac{1}{448} \cdot \frac{z^7}{M^6} + \ldots = z + Mc \quad (6).$$

And putting for x and z their values from (2) we find

$$2p = (2+p)x,\ 2Mp = (2+p)z, \text{ whence } 1+p = \frac{2+x}{2-x} = \frac{2M+z}{2M-z} \quad . \quad (7),$$

And by expanding the first of these equations (7)

$$x = p - \tfrac{1}{2}p^2 + \tfrac{1}{4}p^3 - \tfrac{1}{8}p^4 + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Subtracting (8) from (3), and multiplying by M to find the Mc of (6), we have

$$My - z = Mc = M(\tfrac{1}{12}p^3 - \tfrac{1}{8}p^4 + \tfrac{11}{80}p^5 - \tfrac{13}{96}p^6 + \tfrac{57}{448}p^7 - \ldots) \quad . \quad . \quad (9),$$

a converging series of which the limits are the first term and the first two terms.

Preparation.—To insure p being small in all cases, I have invented the rule of *preparation*, founded on the fact that if N be the number whose logarithm is sought, and a and b any two numbers of which the logarithms are known, such that $Na \div b = n + d$, where n is the next less number to $Na \div b$ in the table, and d, the difference, is less than the difference between two numbers in the table, then $\log N = \log(n+d) + \log b - \log a$. In Tables I and II the difference between two consecutive numbers is ·001, and as there are 100 entries, all the numbers lie between 1 and 1·1; so that if $Na \div b$ is less than 1·1, the required reduction is effected.

Preparation is accomplished in two lines of simple multiplication and division, as follows:—

The given number N is divided or multiplied by such a power of 10 as will leave the quotient or product as a decimal fraction between 1 and 10. This is effected by simply shifting the decimal point.

If the first decimal place is less than 3 times the integer (which is always the case when the integer exceeds 3), divide by the integer and divide the quotient by 1·1 or 1·2. The result is less than 1·1.

If the first decimal place is more than twice the integer, then it is always possible, generally in several ways, to find an integer between 1 and 10 which, used as a multiplier, will give a product of which the integer is less than 13, and the first decimal place less than the integer. The following rule embraces every case:—Multiply any of the numbers 1·30 to 1·340 by 4; 1·340 to 1·80 by 7; 1·80 to 1·960 by 5; 1·960 to 1·99 by 6; 2·50 to 2·99 by 4; 3·80 to 3·99 by 3. Then dividing this product by the integer the quotient is less than 1·1.

This preparation is very convenient also for starting Weddle's and

Hearn's processes given by Mr. Peter Gray in the introduction to his Tables for twelve-place logarithms, 1865 (first published in 1845), and is also very much simpler than that proposed by Mons. Thoman in his "Tables de logarithmes à 27 décimales," 1867.

Interpolation.—The finding of log N is thus made dependent on finding log $(n+d)$, where n is a tabular number and d is less than ·001. We then find $2\text{M}d\div(2n+d)$, which gives the "uncorrected" logarithm of $n+d$, or the "quotient" x or z. The multiplication $2\text{M}\times d$ is effected by the multiples of the bimodulus given in the tables, when M is not 1, the unit place of each multiple of 2M being placed immediately below the determining figure of d, care being taken to preserve as many places as are necessary for the final result. The division is a single contracted division. The resulting x or z has to be "corrected" by the equations (5) and (6), as shown in Section III.

Completion.—Having found log $(n+d)$, we add the logarithm of the power of 10 by which we first divided, and the logarithm of any other divisor, and the arithmetical complement of the logarithm of the power of 10 or any other multiplier. All these logarithms are given in the table. The result is the complete log N to the number of decimal places for which the table is adapted.

Anti-Logarithms.—A logarithm being given we have to reduce it to the logarithm of a number between 1 and 1·1. This is most conveniently done by subtracting from it (or adding to it) the logarithm of the largest power of 10, which will make the result lie between 0 and log 10, and afterwards subtracting the next least logarithm of an integer between 1 and 10, and then the next least logarithm of a number between 1·1 and 2. The logarithms of all these numbers are given in the table. The result will be the logarithm of a number less than 1·1. We then subtract the next less logarithm in the table of interpolation, and obtain the equivalent to the *corrected* quotient $x+c$ or $z+\text{M}c$ of (5) and (6). We find the correction in the same way as for the quotient, and *subtract* it, thus obtaining x or z. Then we divide the bimodulus increased by this x or z, by the bimodulus decreased by this x or z, as in (7), and thus find $1+p$, which is the number corresponding to the "quotient" in the direct method. For "completion" this has to be multiplied by the numbers corresponding to all the logarithms subtracted in the preparation.

Section III.—Calculation of the Bimodular Corrections.

The principal peculiarity of this improved bimodular method consists in the calculation of the corrections and the determination of the number of places which can be trusted in any case, as assigned in the tables.

The repetition of any digit m times within the same number will

be represented by suffixing m to the right of that digit. Thus, $\cdot 0_m 1$ is a decimal fraction beginning with m zeroes and followed by 1, and nat. log $1\cdot 0_7 1 = \cdot 0_8 9_8 5 0_7 3_8 0 8 3_6 5 3_7$ to forty-eight places. Other writers have used 0^m in this sense, but it is not applicable to other digits, and conflicts with the usual notation of powers, thus 230^3 looks like $(230)^3=12{,}167{,}000$, in place of $23{,}000=230_3$.

Write equations (5) and (6) thus—

$$\text{nat. log}\,(1+p)=x+c=x+c_1+c_2+\ldots \quad (10),$$

$$\text{tab. log}\,(1+p)=z+t=z+t_1+t_2+\ldots \quad (11),$$

where

$$c_1=\tfrac{1}{12}x^3=x^3\times\cdot 0833\ldots,\quad c_2=\tfrac{1}{80}x^5=x^5\times\cdot 0125 \quad (12),$$

$$\text{tab. log}\,c_1=3\,\text{tab. log}\,x+\cdot 920\ 8188-2 \quad (13),$$

$$\text{tab. log}\,c_2=5\,\text{tab. log}\,x+\cdot 096\ 9100-2 \quad (14),$$

$$\text{tab. log}\,x=\tfrac{1}{3}\,\text{tab. log}\,c_1+\cdot 359\ 7271 \quad (15),$$

$$t_1=z^3\times\cdot 441\ 824\ 842\ 539\ 87,\quad t_2=z^5\times\cdot 351\ 376\ 544\ 673\ 68. \quad (16),$$

$$\text{tab. log}\,t_1=3\,\text{tab. log}\,z+\cdot 645\ 2501-1 \quad (17),$$

$$\text{tab. log}\,t_2=5\,\text{tab. log}\,z+\cdot 545\ 7728-1 \quad (18),$$

$$\text{tab. log}\,z=\tfrac{1}{3}\,\text{tab. log}\,t_1+\cdot 118\ 2500 \quad (19).$$

By means of these equations the corrections can be calculated from the "quotient" (that is, the approximate values of x and z) either with or without existing tables of logarithms, or the quotient x or z may be calculated to which a particular value of the first correction is due.

From these has been calculated the following table of the critical values of the first and second corrections, upon which the whole practical use of the corrections depends. The quotients were first taken to proceed from $\cdot 0_m 1$ to $\cdot 0_m 9$ by steps of $\cdot 0_m 1$. Then the values of the quotients were determined, which reduced either of the two first corrections to $\cdot 0_n 1$, n being variable, from which point the suffix of 0, or the number of initial zeroes, changed, giving critical values of the corrections. Such quotients were then inserted in numerical order. The approximate numbers were obtained from the quotients on the supposition that p was small enough to make nat. log $(1+p)=p$, and tab. log $(1+p)=\mathrm{M}p$, to three places of significant figures.

The suffix of 0 in the first correction, diminished by 1, shows the number of places which are unaffected by that correction, that is, the number of places in the uncorrected quotient which may be trusted without corrections. The undiminished suffix shows a number of places which cannot be wrong by more than one unit in defect in the

Table of the Critical Values of the Corrections.

I. Natural logarithms.

Approximate number.	Exact quotient.	Value of c_1.	Value of c_2.
$1\cdot 0_m 100$	$\cdot 0_m 100$	$\cdot 0_{3m+4} 833$	$\cdot 0_{5m+6} 125$
106	106	$\cdot 0_{3m+3} 100$	169
152	152	290	$\cdot 0_{5m+5} 100$
200	200	666	400
229	229	$\cdot 0_{3m+2} 100$	786
240	240	116	$\cdot 0_{5m+4} 100$
300	300	225	304
381	381	460	$\cdot 0_{5m+3} 100$
400	400	533	128
493	493	$\cdot 0_{3m+1} 100$	365
500	500	104	391
600	600	180	972
604	604	183	$\cdot 0_{5m+2} 100$
700	700	285	210
800	800	426	409
900	900	607	738

II. Tabular logarithms.

Approximate number.	Exact quotient.	Value of t_1.	Value of t_2.
$1\cdot 0_m 231$	$\cdot 0_m 100$	$\cdot 0_{3m+3} 442$	$\cdot 0_{5m+5} 352$
284	123	827	$\cdot 0_{5m+4} 100$
303	131	$\cdot 0_{3m+2} 100$	137
450	195	329	$\cdot 0_{5m+3} 100$
462	200	353	112
653	282	$\cdot 0_{3m+1} 100$	636
693	300	119	654
713	310	165	$\cdot 0_{5m+2} 100$
925	400	283	360
$1\cdot 0_{m-1} 100$	$\cdot 0_m 434$	$\cdot 0_{3m+1} 361$	$\cdot 0_{5m+2} 542$
113	491	522	$\cdot 0_{5m+1} 100$
116	500	552	110
139	600	954	273
141	609	$\cdot 0_{3m} 100$	295
163	700	152	590
179	778	208	$\cdot 0_{5m} 100$
186	800	226	115
209	900	322	207

last place. The suffix of 0 in the second correction, diminished by 1, shows how many places of the quotient, after applying the first correction, are left unaffected by the second correction, that is, how many places can be trusted on applying the first correction only. For natural logarithms it will be seen that this never gives less than $5m+1$ places, that is, $2m+1$ places in addition to those determined without correction. Thus in Table I, where m is never less than 3, we can always obtain sixteen places. For tabular logarithms, as in Table II, we must first observe a critical value in the numbers themselves. In that table the number $1+p$, whose logarithm is finally sought, must be less than 1·001. Hence, while in the upper part of the preceding table of critical values, m will always be 3 or more, in the lower part, $m-1$ will always be 3 or more, so that m will always be 4 or more. As far then as the quotient $\cdot 0_3 434$, the first correction gives only $5.3+1=16$ places, and this is the largest quotient that can commence with $\cdot 0_3$. If the significant figures are greater than 434, then m will be 4, and up to the quotient $\cdot 0_4 778$ we can trust $5.4=20$ places, and beyond it we can even trust 19 places. Observe that $\cdot 0_4 9$ at the bottom of this table is followed by $\cdot 0_3 1$ at the top (II, second column), for which, also, the second corrections leave $5.3+4=19$ places unaffected.

But in determining the full number of places of the first correction from the uncorrected quotient by equations (12) and (16), we are, of course, obliged to take so many significant places, that on cubing the result and multiplying by the proper coefficient, no error affecting the full number of places should be committed. The number of places required for this purpose is so large that if we calculated the result directly, the present method of correction would be illusory. Hence it is necessary to use common seven-place logarithmic tables which can be trusted to six places. Consequently, we can use only six significant places in the quotient for finding the correction, and we thus introduce an error not exceeding half a unit in the last place in excess or defect. On estimating the limiting effect of this error, I find practically that on using six significant places of the uncorrected quotient to determine the first correction, we may trust all six places of the correction found. The total number of places that can be trusted, when this error is allowed for, depends on the quotient. Let r be the significant places of the quotient converted into a decimal fraction with one unit place. Then the real quotient is $\cdot 0_m 1 \times r$, but on taking only six significant places, we use as a quotient $\cdot 0_m 1 \times r \pm \cdot 0_{m+6} 1 \times 5$, and the error thus made in the correction may be taken as the term involving r^2 in the cube of this number divided by $12M^2$, that is, as $\cdot 0_{3m+8} 1 \times 15r^2 \div 12M^2$. Then, putting $15r^2 \div 12M^2 = 10$ and 100, and finding the corresponding values of r, we obtain the critical values of the quotient where the suffix of 0 in the error of the

Bimodular Table I.—Natural Logarithms.

1. Table for Interpolation.

No.	Natural Logarithm.	No.	Natural Logarithm.
1·000	·000 000 000 000 000 000	1·050	·048 790 164 169 432 003
1	·000 999 500 333 083 533	51	·049 742 091 894 814 074
2	·001 998 002 662 673 056	52	·050 693 114 315 518 118
3	·002 995 508 979 798 478	53	·051 643 233 151 838 450
4	·003 992 021 269 537 453	54	·052 592 450 119 170 584
1·005	·004 987 541 511 039 074	1·055	·053 540 766 928 029 818
6	·005 982 071 677 547 464	56	·054 488 185 284 069 731
7	·006 975 613 736 425 242	57	·055 434 706 888 100 582
8	·007 968 169 649 176 874	58	·056 380 333 436 107 639
9	·008 959 741 371 471 904	59	·057 335 066 619 269 407
1·010	·009 950 330 853 168 083	1·060	·058 268 908 123 975 776
11	·010 939 940 038 334 364	61	·059 211 859 631 846 083
12	·011 928 570 865 273 802	62	·060 153 922 819 747 091
13	·012 916 225 266 546 328	63	·061 095 099 359 810 876
14	·013 902 905 168 991 421	64	·062 035 390 919 452 641
1·015	·014 888 612 493 750 655	1·065	·062 974 799 161 388 435
16	·015 873 349 156 290 149	66	·063 913 325 743 652 797
17	·016 857 117 066 422 899	67	·064 850 972 319 616 314
18	·017 839 918 128 331 000	68	·065 787 740 538 003 097
19	·018 821 754 240 587 761	69	·066 723 632 042 908 173
1·020	·019 802 627 296 179 713	1·070	·067 658 648 473 814 805
21	·020 782 539 182 528 504	71	·068 592 791 465 611 716
22	·021 761 491 781 512 692	72	·069 526 062 648 610 245
23	·022 739 486 969 489 429	73	·070 458 463 648 561 419
24	·023 716 526 617 316 042	74	·071 389 996 086 672 945
1·025	·024 692 612 590 371 501	1·075	·072 320 661 579 626 121
26	·025 667 746 748 577 792	76	·073 250 461 739 592 673
27	·026 641 930 946 421 178	77	·074 179 398 174 251 512
28	·027 615 167 032 973 365	78	·075 107 472 486 805 412
29	·028 587 456 851 912 555	79	·076 034 686 275 997 608
1·030	·029 558 802 241 544 403	1·080	·076 961 041 136 128 325
31	·030 529 205 034 822 873	81	·077 886 538 657 071 225
32	·031 498 667 059 370 991	82	·078 811 180 424 289 778
33	·032 467 190 137 501 495	83	·079 734 968 018 853 559
34	·033 434 776 086 237 388	84	·080 657 903 017 454 467
1·035	·034 401 426 717 332 396	1·085	·081 579 986 992 422 874
36	·035 367 143 837 291 316	86	·082 501 221 511 743 696
37	·036 331 929 247 390 277	87	·083 421 608 139 072 391
38	·037 295 784 743 696 896	88	·084 341 148 433 750 885
39	·038 258 712 117 090 341	89	·085 259 843 950 823 419
1·040	·039 220 713 153 281 296	1·090	·086 177 696 241 052 332
41	·040 181 789 632 831 832	91	·087 094 706 850 933 767
42	·041 141 943 331 175 177	92	·088 010 877 322 713 299
43	·042 101 176 018 635 394	93	·088 926 209 194 401 509
44	·043 059 489 460 446 977	94	·089 840 703 999 789 463
1·045	·044 016 885 416 774 327	1·095	·090 754 363 268 464 143
46	·044 973 365 642 731 158	96	·091 667 188 525 823 792
47	·045 928 931 888 399 803	97	·092 579 181 293 093 194
48	·046 883 585 898 850 420	98	·093 490 343 087 338 889
49	·047 837 329 414 160 123	99	·094 360 675 421 484 311

2. *For Preparation.*

No.	Natural Logarithm.					
1·1	0·095	310	179	804	324	869
1·2	0·182	321	556	793	954	626
1·3	0·262	364	264	467	491	052
1·4	0·336	472	236	621	212	931
1·5	0·405	465	108	108	164	382
1·6	0·470	003	629	245	735	554
1·7	0·530	628	251	062	170	396
1·8	0·587	786	664	902	119	008
1·9	0·641	853	886	172	394	776
2·0	0·693	147	180	559	945	309
3·0	1·098	612	288	668	109	691
4·0	1·386	294	361	119	890	619
5·0	1·609	437	912	434	100	375
6·0	1·791	759	469	228	055	001
7·0	1·945	910	149	055	313	305
8·0	2·079	441	541	679	835	928
9·0	2·197	224	577	336	219	383
10·0	2·302	585	092	994	045	684
11·0	2·397	895	272	798	370	544
12·0	2·484	906	649	788	000	310

3. *Multiples of nat. log* 10.

No. of mult.	Natural Logarithm.					
1	2·302	585	092	994	045	684
2	4·605	170	185	988	091	368
3	6·907	755	278	982	137	052
4	9·210	340	371	976	182	736
5	11·512	925	464	970	228	420
6	13·815	510	557	964	274	104
7	16·118	095	650	958	319	788
8	18·420	680	743	952	365	472
9	20·723	265	836	946	411	156
10	23·025	850	929	940	456	840
11	25·328	436	022	934	502	524
12	27·631	021	115	928	548	208
13	29·933	606	208	922	593	892
14	32·236	191	301	916	639	576
15	34·538	776	394	910	685	260
16	36·841	361	487	904	730	944

4. *For no Corrections.*

For Difference, or Quotient,	Trust places, uncorrected,	
$\cdot0_{2}100$	9	And one place more in each case with a probable error in it of one unit in defect.
$\cdot0_{3}493$	10	
$\cdot0_{3}229$	11	
$\cdot0_{3}106$	12	
$\cdot0_{4}493$	13	
$\cdot0_{4}229$	14	
$\cdot0_{4}106$	15	
$\cdot0_{5}193$	16	

For intermediate quotients trust the number of places opposite the next greater in the above table.

5. *For Full Corrections, Additive.*

Take six significant figures of the quotient, and use six significant figures of the cor. from this formula—

tab. log cor. = 3 tab. log quotient + ·920 8188 − 2.

Trust all the places thus corrected, that is—

For Difference, or Quotient,	Trust in result places,	
$\cdot0_{2}100$	14	And one place more in each case with a probable error in it of one unit in defect.
$\cdot0_{3}894$	15	
$\cdot0_{3}284$	16	
$\cdot0_{3}100$	17	
$\cdot0_{4}894$	18	

For intermediate quotients trust the number of places opposite the next greater.

6. *For Short Corrections, Additive, giving twelve places.*

Work to thirteen places. Possible error on "completion" one unit in the twelfth place. For intermediate quotients use the correction opposite the next less.

Quotnt.	Cor.	Quotnt.	Cor.	Quotnt.	Cor.
$\cdot0_{3}000$	$\cdot0_{10}00$	$\cdot0_{3}707$	$\cdot0_{10}30$	$\cdot0_{3}894$	$\cdot0_{10}60$
182	1	715	31	899	61
262	2	723	32	904	62
311	3	731	33	909	63
348	4	738	34	913	64
$\cdot0_{3}378$	$\cdot0_{10}05$	$\cdot0_{3}745$	$\cdot0_{10}35$	$\cdot0_{3}918$	$\cdot0_{10}65$
404	6	752	36	923	66
427	7	759	37	928	67
448	8	766	38	932	68
467	9	773	39	937	69
$\cdot0_{3}485$	$\cdot0_{10}10$	$\cdot0_{3}780$	$\cdot0_{10}40$	$\cdot0_{3}941$	$\cdot0_{10}70$
501	11	786	41	946	71
517	12	792	42	950	72
531	13	798	43	952	73
545	14	805	44	959	74
$\cdot0_{3}558$	$\cdot0_{10}15$	$\cdot0_{3}811$	$\cdot0_{10}45$	$\cdot0_{3}963$	$\cdot0_{10}75$
571	16	817	46	968	76
583	17	823	47	972	77
594	18	829	48	976	78
606	19	835	49	980	79
$\cdot0_{3}616$	$\cdot0_{10}20$	$\cdot0_{3}841$	$\cdot0_{10}50$	$\cdot0_{3}984$	$\cdot0_{10}80$
627	21	846	51	989	81
637	22	852	52	993	82
646	23	857	53	997	83
656	24	863	54		
$\cdot0_{3}665$	$\cdot0_{10}25$	$\cdot0_{3}868$	$\cdot0_{10}55$		
674	26	873	56		
683	27	879	57		
691	28	884	58		
699	29	889	59		

Bimodular Table II.—Tabular Logarithms.

1. Table for Interpolation.

No.	Tabular Logarithm.	No.	Tabular Logarithm.
1·000	·000 000 000 000 000 000	1·050	·021 189 299 069 938 073
1	·000 434 077 479 318 641	51	·021 602 716 028 242 220
2	·000 867 721 531 226 912	52	·022 015 739 817 720 259
3	·001 300 933 020 418 119	53	·022 428 371 185 486 518
4	·001 733 712 809 000 530	54	·022 840 610 876 527 803
1·005	·002 166 061 756 507 676	1·055	·023 252 459 633 711 470
6	·002 597 980 719 908 592	56	·023 663 918 197 793 454
7	·003 029 470 553 618 007	57	·024 074 987 307 426 268
8	·003 460 532 109 506 486	58	·024 485 667 699 166 953
9	·003 891 166 236 910 522	59	·024 895 960 107 485 003
1·010	·004 321 373 782 642 574	1·060	·025 305 865 264 770 241
11	·004 751 155 591 001 063	61	·025 715 383 901 340 666
12	·005 180 512 503 780 310	62	·026 124 516 745 450 260
13	·005 609 445 360 280 428	63	·026 533 264 523 296 757
14	·006 037 954 997 317 171	64	·026 941 627 959 029 378
1·015	·006 466 042 249 231 723	1·065	·027 349 607 774 756 528
16	·006 893 707 947 900 450	66	·027 757 204 690 553 459
17	·007 320 952 922 744 597	67	·028 164 419 424 469 893
18	·007 747 778 000 739 942	68	·028 571 252 692 537 612
19	·008 174 184 006 426 395	69	·028 977 705 208 778 017
1·020	·008 600 171 761 917 561	1·070	·029 383 777 685 209 641
21	·009 025 742 086 910 247	71	·029 789 470 831 855 634
22	·009 450 895 798 693 927	72	·030 194 785 356 751 215
23	·009 875 633 712 160 158	73	·030 599 721 965 951 084
24	·010 299 956 639 811 952	74	·031 004 281 363 536 802
1·025	·010 723 865 391 773 104	1·075	·031 408 464 251 624 136
26	·011 147 360 775 797 468	76	·031 812 271 330 370 371
27	·011 570 443 597 278 197	77	·032 215 703 297 981 585
28	·011 993 114 659 256 928	78	·032 618 760 850 719 897
29	·012 415 374 762 432 929	79	·033 021 444 682 910 673
1·030	·012 837 224 705 172 205	1·080	·033 423 755 486 949 702
31	·013 258 665 283 516 547	81	·033 825 693 953 310 343
32	·013 679 697 291 192 549	82	·034 227 260 770 550 632
33	·014 100 321 519 620 579	83	·034 628 456 625 320 360
34	·014 520 538 757 923 700	84	·035 029 282 202 368 120
1·035	·014 940 349 792 936 558	1·085	·035 429 738 184 548 315
36	·015 359 755 409 214 218	86	·035 829 825 252 828 143
37	·015 778 756 389 040 962	87	·036 229 544 086 294 540
38	·016 197 353 512 439 047	88	·036 628 895 362 161 100
39	·016 615 547 557 177 412	89	·037 027 879 755 774 956
1·040	·017 033 339 298 780 355	1·090	·037 426 497 940 623 635
41	·017 450 729 510 536 156	91	·037 824 750 588 341 878
42	·017 867 718 963 505 669	92	·038 222 638 368 718 428
43	·018 284 308 426 530 869	93	·038 620 161 949 702 792
44	·018 700 498 666 243 352	94	·039 017 321 997 411 969
1·045	·019 116 290 447 072 807	1·095	·039 414 119 176 137 143
46	·019 531 684 531 255 434	96	·039 810 554 148 350 354
47	·019 946 681 678 842 334	97	·040 206 627 574 711 132
48	·020 361 282 647 707 846	98	·040 602 340 114 073 104
49	·020 775 488 193 557 860	99	·040 997 692 423 490 567

Bimodular Table II.—Tabular Logarithms—*continued.*

2. *For Preparation.*

No.	Tabular Logarithm.
1·1	0·041 392 685 158 225 041
1·2	0·079 181 246 047 624 828
1·3	0·113 943 352 306 836 769
1·4	0·146 128 035 678 238 026
1·5	0·176 091 259 055 681 242
1·6	0·204 119 982 655 924 781
1·7	0·230 448 921 378 273 929
1·8	0·255 272 505 103 306 070
1·9	0·278 753 600 952 828 962
2·0	0·301 029 995 663 981 195
3·0	0·477 121 254 719 662 437
4·0	0·602 059 991 327 962 390
5·0	0·698 970 004 336 018 805
6·0	0·778 151 250 383 643 633
7·0	0·845 098 040 014 256 831
8·0	0·903 089 986 991 943 586
9·0	0·954 242 509 439 324 875
10 0	1·000 000 000 000 000 000
11·0	1·041 392 685 158 225 041
12·0	1·079 181 246 047 624 827

3. *Multiples of the Bimodulus.*

No. of mult.	Value of Multiple.
1	0·868 588 963 806 503 655
2	1·737 177 927 613 007 311
3	2·605 766 891 419 510 966
4	3·474 355 855 226 014 621
5	4·342 944 819 032 518 277
6	5·211 533 782 839 021 932
7	6·080 122 746 645 525 587
8	6·948 711 710 452 029 242
9	7·817 300 674 258 532 898

4. *For no Corrections.*

Difference.	Quotnt.	Trust Places	
$\cdot 0_2 100$	$\cdot 0_3 434$	9	And one more place in each case with a probable error in it of one unit in defect.
$\cdot 0_3 653$	$\cdot 0_3 282$	10	
$\cdot 0_3 303$	$\cdot 0_3 131$	11	
$\cdot 0_3 141$	$\cdot 0_4 609$	12	
$\cdot 0_4 653$	$\cdot 0_4 282$	13	
$\cdot 0_4 303$	$\cdot 0_4 131$	14	
$\cdot 0_4 141$	$\cdot 0_5 609$	15	
$\cdot 0_5 653$	$\cdot 0_5 282$	16	

For intermediate Differences and Quotients trust the number of places opposite the next greater.

5. *For Full Corrections, Additive.*

Take six significant figures of the quotient and use six significant figures of the correction from this formula—

tab. log cor. = 3 tab. log quotient + ·645 2501 − 1.

Difference.	Quotnt.	Trust places	
$\cdot 0_2 100$	$\cdot 0_3 434$	14	And one more place in each case with a probable error in it of one unit in defect.
$\cdot 0_3 893$	$\cdot 0_3 388$	15	
$\cdot 0_3 284$	$\cdot 0_3 123$	16	
$\cdot 0_3 231$	$\cdot 0_3 100$	17	
$\cdot 0_4 893$	$\cdot 0_4 388$	18	

For intermediate Differences and Quotients trust the number of places opposite the next greater.

6. *For Short Corrections, Additive, giving twelve places, with a possible error of one unit in the twelfth place on completion.*

Quotient.	Correction.	Quotient.	Correction.
$\cdot 0_3 000$	$\cdot 0_{10} 00$	$\cdot 0_3 353$	$\cdot 0_{10} 20$
104	01	359	21
150	02	365	22
178	03	371	23
199	04	376	24
$\cdot 0_3 217$	$\cdot 0_{10} 05$	$\cdot 0_3 381$	$\cdot 0_{10} 25$
232	06	386	26
245	07	391	27
257	08	396	28
268	09	401	29
$\cdot 0_3 278$	$\cdot 0_{10} 10$	$\cdot 0_3 406$	$\cdot 0_{10} 30$
288	11	410	31
296	12	415	32
305	13	419	33
313	14	423	34
$\cdot 0_3 320$	$\cdot 0_{10} 15$	$\cdot 0_3 427$	$\cdot 0_{10} 35$
327	16	432	36
334	17		
341	18		
348	19		

For intermediate quotients take the correction opposite the next less.

Note.—The natural logarithms to eighteen places in Table I are either taken direct or calculated (by subtracting nat. logs of 1,000 and 10) from "Wolframii Tabula Logarithmorum Naturalium" to forty-eight places, appended to Vega's "Thesaurus Logarithmorum Completus," 1794.
The tabular logarithms to eighteen places in Table II are taken direct from Mr. Peter Gray's "Tables for the formation of Logarithms and Anti-Logarithms to twenty-four places," 1876.
In both tables the arrangement and corrections are original.

correction changes. The results are given under "5. Full Corrections," in Tables I and II.

But although it is by no means difficult or very troublesome to use the formulæ (13) and (17) for finding the first correction, it is always inconvenient to use two tables. It would be manifestly impossible to give a table of corrections to six figures within reasonable limits. Hence, leaving the "full correction" to be found, when desired, by these formulæ, I append a table of "short corrections," so as to obtain twelve places of the result from Tables I and II at sight. The thirteenth place has been allowed for, so that the result may be thoroughly trusted, but in the "completion" an error of one unit in the twelfth place may easily creep in unless "full corrections" are used. These "short corrections" have been calculated from the formulæ (15) and (19), by assuming successive values of the first correction, as $\cdot 0_{12}5$, $\cdot 0_{11}15$, $\cdot 0_{11}25$ and so on, and calculating the corresponding value of the quotients. But in the table itself these corrections are entered as $\cdot 0_{11}1$, $\cdot 0_{11}2$, &c. The limiting correction is reached when the corresponding quotient is the next least to that due to the number 1·001. These twelve places are fully as many as are required for ordinary purposes, and for them only thirteen out of the eighteen places in the tables should be used.

Section IV.—Bimodular Tables and Examples.

Table I applies to natural logarithms giving from nine to sixteen places, according to circumstances, with no corrections, twelve places with short corrections, and fourteen to sixteen places with full corrections.

Rule to find the logarithm from the number. — Reduce the given number to the form of a decimal fraction with an integer less than 10.

Multiply and divide by such whole numbers less than 13 as will reduce the number to one less than 1·1, as shown in Section II.

Find the next less number in "1. Table for Interpolation," and first subtract it from the reduced number, then omit the decimal point, and multiply by 2, forming the "dividend;" secondly, add this next less number to the reduced number, and then omit the decimal point, forming the "divisor."

Divide the dividend by the divisor by simple contracted division to as many places as are required. Correct the quotient, as may be necessary, by the table or formula of correction, No. 6 or 5.

Add the logarithms of the divisors and the arithmetical complements of the logarithms of the multipliers used in forming the reduced number, to find the full corrected logarithm.

Table II applies to Briggs's or Tabular Logarithms, giving from nine to sixteen places, according to circumstances, with no correc-

tions, twelve places with short corrections, and fourteen to eighteen places with full corrections.

Rule.—Proceed precisely as for natural logarithms, except that instead of multiplying by 2 it is necessary to multiply by the tabular bimodulus, by help of the multiples given in No. 3.

Tables I and II. *Rule to find the number from the logarithm.*—Subtract the logarithm of the next lower power of 10, and then, in order, the next lower logarithm in the lower, and then that in the upper part of the table "2. For Preparation," and afterwards the next lower logarithm in the table for interpolation.

Considering this as an approximate logarithm of a reduced number, find the correction as if it were a quotient by No. 5 or 6, and *subtract* (instead of adding) the correction, which reduces it to the form of a quotient or approximate logarithm.

Add the resulting number to and subtract it from the bimodulus (which is 2 for natural logarithms) and divide the sum by the difference.

Multiply the quotient in succession by the numbers corresponding to the logarithms subtracted. The result is the number required.

Examples, fully worked out, with explanations.

Let $$N=192\ 699\ 928\ 576=(76)^6.$$

Then calculating the value of 6 nat. log 76 from Wolfram's tables appended to Vega's, and multiplying the result by the tabular modulus we find to twenty places—

$$\text{nat. log } N=25{\cdot}984\ 400\ 041\ 717\ 986\ 473\ 06$$

$$\text{tab. log } N=11{\cdot}284\ 881\ 553\ 684\ 748\ 111\ 78$$

These numbers serve as checks to the correctness of the following work.

Here a, b, c form the "preparation" of N. As a begins with 1·9, where the first decimal place is more than 3 times the integer, a is multiplied by 6 to produce 11·56 . . ., a decimal fraction of which the integer 11 is less than 13 and more than twice the first integer 5. Both 5 and 4 would have also answered. The divisor 11 is separated off by), and in the quotient c the next less number 1·051 in the table for interpolation is similarly separated. This leaves $c-1{\cdot}051$ to the right of), with the decimal point already omitted. Then this difference is multiplied by the bimodulus 2, to obtain the dividend d. The whole of c is added to the separated part 1·051, and then the decimal point is omitted, giving e. As the difference $c={\cdot}0_4905$, lies between ${\cdot}0_3106$ and ${\cdot}0_4493$, we can certainly obtain twelve places without correction (Table I, No. 4), and as it lies between ${\cdot}0_31$ and ${\cdot}0_4894$ we can obtain seventeen places with full corrections (Table I, No. 5). We

Ex. 1. To Table I.—Find nat. log N to sixteen places. The letters refer to the following explanations. Every figure required by the most moderate calculator is inserted.

	$a=\mathrm{N}\div10^{11}$.	1·92 699 928 576	*a*
	$b=6a$.	11) ·56 199 571 456	*b*
	$c=b\div11$.	1·05 1)09 051 950 545 454 54	*c*
e	210 209 051 950 545 454 54	)18 103 901 090 909 09	*d*
	· ··· ··· ··· ··· ———	16 816 724 156 043 63	(8
f	·000 086 123 318 301 099	1 287 176 934 865 46	
g	$\cdot0_{13}$ 53 233	1 261 254 311 703 27	(6
h	·049 742 091 894 814 074	25 922 623 162 69	
k	2·397 895 272 798 370 544	21 020 905 195 05	(1
l	8·208 240 530 771 944 999−10	4 901 717 967 14	
m	25·328 436 022 934 502 524	4 204 181 039 01	(2
n	25·984 400 041 717 986 473	697 536 928 13	
		630 627 155 90	(3
	$d=2(c-1{\cdot}051)\times10^{19}$=dividend.	66 909 772 23	
	$e=2(c+1{\cdot}051)\times10^{19}$=divisor.	63 062 715 59	(3
	$f=d\div e$=quotient.	3 847 056 64	
	g=full correction, see below.	2 102 090 64	(2
	h=nat. log 1·051.	1 744 966 12	
	k=nat. log 11.	1 681 672 41	(8
	l=arithm. comp. of nat. log 6.	63 293 71	
	m=11 nat. log 10.	63 062 72	(3
	n=nat. log N, true to 18 places.	230 99	
		210 21	(01
	Calculation of g. Log f, taking six significant places,	20 78	
	=log $\cdot0_4861$ 233= ·935 1206 − 5	18 92	(09
	3 log f=2·805 3618 −15	1 86	
	+ ·920 8188 − 2	1 89	(9
	log g=log $\cdot0_{13}532$ 329= ·726 1806 −14		

prepare, then, for seventeen places, by carrying the quotient c far enough to allow of obtaining eighteen places, that is, fourteen significant places of the quotient f. As at least 2 digits of the divisor must remain for the last contracted divisor, we shall require only fifteen places of the divisor, and the last five are rejected (shown by drawing a line under them). The successive digits of the quotient are written to the right after ((following Briggs's use), and are collected in f. The rest of the process is evident from the notes

made. The result happens to be correct to eighteen places, in place of the guaranteed seventeen; but this is quite accidental, as the last or eighteenth place of all the logarithms used is always in excess or defect.

Ex. 2. To Table II.—Find tab. log N to twelve places by the short corrections.

$a=\mathrm{N}\div10^{11}$.	1 ·92 699 928 576	a
$b=5a$.	9) ·63 499 642 880	b
$c=b\div9$.	1 ·07 0)55 515 875 556	c
$d=5\times$ bimodulus $\times10^{10}$.	43 429 448 190	d
$5\times$,, $\times10^{9}$.	4 342 944 819	
$5\times$,, $\times10^{8}$.	434 294 482	
$e=1\times$,, $\times10^{7}$.	08 685 890	e
$5\times$,, $\times10^{6}$.	4 342 945	
$8\times$,, $\times10^{5}$.	694 871	
$7\times$,, $\times10^{4}$.	60 801	
$5\times$,, $\times10^{3}$.	4 343	
$5\times$,, $\times10^{2}$.	434	
$5\times$,, $\times10$.	43	
$6\times$,,	5	

g	214 055 515 875 556)	48 220 476 823	f
	· ··· ··· ··	42 811 103 175	(2
h	·000 225 270 891 2	5 409 373 648	
k	$\cdot0_{11}$ 5	4 281 110 318	(2
m	·029 383 777 685 2	1 128 263 330	
n	·954 242 509 439 3	1 070 277 579	(5
p	·301 029 995 664 0−1	57 985 751	
q	11 ·0	42 811 103	(2
r	11 ·284 881 553 684 7	15 174 648	
		14 983 886	(7
		190 762	
		171 244	(08
		19 518	
		19 265	(9
		253	
		214	(1
		39	
		40	(2

$f=(c-1\cdot070)\times10^{14}\times$ bimodulus $=$ dividend.
$g=(c+1\cdot070)\times10^{14}=$ divisor.
$h=f\div g=$ quotient.
$k=$ short correction for quotient ·000 225.
$m=$ tab. log 1 ·070.
$n=$ tab. log 9.
$p=$ arithm. comp. of tab. log 5.
$q=11$ tab. log 10.
$r=$ tab. log N, correct to 13 places.

Ex. 3. To Table II. Given the tab. log N, to eighteen places of decimals, to find N to the greatest possible number of digits. This process is entirely new, and depends upon Section II, eq. (7).

a=tab. log N.	11·284 881 553 684 748 112	a
b=tab. log 10^{11}+tab. log 1·9.	11·278 753 600 952 828 962	b
$c=a-b$.	·006 127 952 731 919 150	c
d=tab. log 1·014.	·006 037 954 997 317 171	d
tab. log $e=c-d$.	·000 089 997 734 601 979	tab. log e
f=correction, see below.	$·0_{12}$ 322 066	f
tab. log e'=tab. log $e-f$.	·000 089 997 734 279 913	tab. log e'
h=bimodulus.	·868 588 963 806 503 655	h

l ·868 498 966 072 223 742)	·868 678 961 540 783 568	k
· ··· ··· ··· ···	868 498 966 072 223 742	(1
e 1·000 207 248 915 1(8(7 4	179 995 468 559 826	
m 10 002 072 489 151 9	173 699 793 214 445	(0002
n 4 000 828 995 660 7	6 295 675,345 381	
p 1·014 210 150 400 000 (0	6 079 492 762 505	(07
q ·912 789 135 360 000 0	216 182 582 876	
N 1926 999 285 76·0 000 0	173 699 793 214	(2
	42 482 789 662	
$k=h$+tab. log e'.	34 739 958 643	(4
$l=h-$tab. log e'.	7 742 831 019	
$e=k\div l$.	6 947 991 728	(8
$m=e\times$ ·01.	794 839 291	
$n=e\times$ ·004.	781 649 069	(9
$p=e+m+n=e\times 1·014$.	13 190 222	
$q=p\times$ ·09.	8 684 990	(1
N=$(p+q)·10^{11}=p\times 1·09\times 10^{11}$.	4 505 232	
	4 342 495	(5
Calculation of f—	162 737	
r=tab. log e= $·0_4899$ 977 taken as quotient in	86 850	(1
s=tab. log r= ·954 2314− 5 Table II, No. 5.	75 887	
$3s$=2·862 6942−15	69 480	(8
t= ·645 2501− 1	6 407	
$3s+t$=log f= ·507 9443−13	6 079	(7
	328	
	260	(3
	68	

The preparation a, b, c, is similar to that in Ex. 1, but 5 is used as the multiplier by way of variety. The difference $c-1{\cdot}070$ being $\cdot0_355$ . . ., which lies between $\cdot0_3653$ and $\cdot0_3303$, we cannot be certain of more than ten places without correction (Table II, No. 4). As only twelve places are wanted, we use the short corrections and work to thirteen places. The chief peculiarity relates to the multiplication of $c-1{\cdot}070$ by the bimodulus by means of the multiples in Table II, No. 3, omitting all the decimal points. The integer of the multiple is placed under the determining figure of the multiplicand, and the multiple is then written out as far as necessary, neglecting the point, but regulating the last figure. It is best to write in the integer 0, as in line e, to preclude error. As the quotient must begin with $\cdot0_3$, only ten significant places are wanted, and hence only eleven places in the divisor g, the four underlined 5556 are therefore rejected. The correction k is found from Table II, No. 6, as belonging to a quotient between $\cdot0_3217$ and 0_3232. The rest is sufficiently explained in the notes. The result is accidentally correct to thirteen places.

Here a is the given tab. log to eighteen places. We first subtract 11 log 10, or the characteristic. Next, if the remainder were greater than any logarithm in the lower part of Table II, No. 2, we should subtract that. But in this case it is not, and hence we proceed to the upper part of No. 2, and subtract the next less, or tab. log 1·9. This completes the preparation, as the difference $c=a-b$, lies between the tab. logs of 1·014 and 1·015 in No. 1, the table for interpolation. Hence, subtracting tab. log 1·014, we find tab. log e, of which the number e has to be found. Now, the formula (7) applies only to an uncorrected $z=$tab. log e', which cannot differ from tab. log e in the three first significant figures. In the direct process, tab. log e is found from tab. log e' by adding the correction found by Table II, No. 5. Hence we have only to subtract this correction f, which is calculated from the same first six significant figures in both cases, as shown in the example. Having found this uncorrected tab. log e', we add it to and subtract it from, the bimodulus, obtaining k and l respectively, and thus find $e=k\div l$. Now, tab. log e' cannot be greater than the greatest difference between two tab. logs in Table II, No. 1, "for interpolation," that is, it cannot be greater than ·000 434 077 . . ., and hence than $\cdot001\times$ modulus. Hence the result of this division $k\div l$, must be less in any case than $(2\text{M}+\cdot001\times\text{M})\div(2\text{M}-\cdot001\times\text{M})=1{\cdot}0010005$ . . . , and must be greater than 1, hence it must commence with 1·000. As the modulus divides out, this conclusion holds for all systems of logarithms. As the last divisor in the contracted division must have two digits for safety, it follows that the number of digits in the quotient $k\div l=e$, will be one less than the number of digits in the divisor, that is, than the number of decimal places in the given logarithm. And as the first of these digits is a whole number,

it follows that the number of decimal places in the quotient $k+l$, will be two less than in the given logarithm. Moreover, as the last decimal place is always approximate, it follows that the number e cannot be found with certainty to more than three decimal places less than the number of decimal places in the given tab. logarithm. Hence, in the present case, although tab. log e is known to eighteen places of decimals, e is known with certainty only to fifteen places of decimals (and sixteen digits). But the error in the next place (or digit) will not probably exceed one unit.

Having found e, we have to multiply it in succession by the numbers corresponding to the logarithms subtracted in the preparations in this example, 1·014, 1·9, and 10^{11}. This is most readily done in the way sufficiently explained by the notes in the example. The resulting number is accidentally correct to seventeen digits, but only sixteen can be used with certainty. Hence, if we use this bimodular method of finding logarithms and anti-logarithms, we should always find the logarithms to two or three places of decimals more than we require digits in the final number to be found.

V. "On the Potential Radix as a Means of Calculating Logarithms to any Required Number of Decimal Places, with a Summary of all Preceding Methods Chronologically Arranged." By Alexander J. Ellis, B.A., F.R.S., F.S.A. Received January 17, 1881.

In the tables attached to my paper "On an Improved Bimodular Method of Computing Logarithms, &c." ("Proc. Roy. Soc.," vol. 31, p. 381), the logarithms used were all taken direct, or immediately calculated, from the tables of Wolframm and Gray. But a complete method of calculating logarithms should be independent of extraneous aid and be applicable to the first construction of tables of logarithms. I shall here show that my improved bimodular method is capable of furnishing a practical means of calculating natural logarithms, and hence logarithms to any base and to any number of places of decimals.

By the term *positive numerical radix* I shall understand a table of the numbers r, $1{\cdot}r$, $1+{\cdot}0_m r$, with their corresponding natural logarithms, where r varies from 1 to 9, 0_m means a series of m zeroes, and m varies from 1 to any required number. The word *Radix* in this sense is adopted from R. Flower, 1771, mentioned below. By the term *negative numerical radix* I mean a similar table of $1-{\cdot}0_m r$, and the negatives of their corresponding logarithms. When these radixes (forming an English plural, as *radices* would be misleading) have been